APPLICATION OF NON-INTEGRAL METHOD TO COMBUSTION AND NATURAL CONVECTION PROBLEMS

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A straightforward, approximate "non-integral" method has been introduced and used to solve the governing differential equations of transport problems with certain assumed profiles for the dependent variables. The method also requires that the integrals of the square of the errors incurred by attempting to satisfy the governing equations be minimized. The methodology is illustrated by determining (i) the burning rate of a fuel surface subject to forced convection, and (ii) the natural convective heat transfer on a heated vertical flat surface. The accuracy of this non-integral method is comparable to that of the conventional integral method. When the non-integral method is modified slightly (by setting the total error equal to zero), the solution is identical to the integral solution.

KEYWORDS Non-integral method Fuel combustion Natural convection

1. INTRODUCTION

One of the conventional approaches to solving the conservation equations for transport processes is the integral method. The integral method has been applied to problems involving forced and free convection, heat transfer from fins, the melting of ice, etc. It has also been used to analyze the combustion of fuels under natural convection, forced convection, and stagnation flow conditions (Annamalai and Sibulkin, 1978). The method requires the use of the integral form of the appropriate conservation equations. The complexity increases for coupled problems such as in natural convection, where the solution of the momentum equation is dependent upon the solution of the energy conservation equation. One has to solve two first-order differential equations for the boundary layer thickness and the maximum flow velocity (Eckert and Drake, 1959).

Recently, Annamalai et al. (1980a) introduced the "non-integral" method, which uses the governing differential equations directly and requires that the integrals of the square of the error in satisfying the differential equations with assumed profiles for the dependent variables be minimized. Three examples—one-dimensional conduction in a fin, forced convection from a flat plate in parallel flow, and phase change in a semi-infinite solid, were used to validate the method. The accuracy of the non-integral technique was demonstrated to be comparable to that of conventional techniques.
In the present paper, the application of the non-integral technique is illustrated by determining (i) the burning rate of a fuel surface subject to forced convection and (ii) the natural convective heat transfer on a heated vertical flat surface. Furthermore, it is found that solutions identical to the integral solutions in both cases can be obtained by slightly modifying the non-integral method.

2. **COMBUSTION SUBJECT TO FORCED CONVECTION**

The procedure of using the non-integral method to analyze the burning rate of a flat fuel surface with a forced convective flow of air over it is now presented. The results of the analysis are applicable to the combustion of plastics, carbon, and liquid fuels. Figure 1 shows the schematic of such a combustion process subject to forced convection.

2.1 **Idealizations†**

The following idealizations are made for the analysis: (a) the flow is two-dimensional and steady; (b) the boundary layer approximation and Fick's law are applicable; (c) the product of density and diffusion coefficient is a constant; (d) viscous dissipation is negligible; and (e) Schmidt number is equal to 1.0.‡

2.2 **Governing Equations and Boundary Conditions**

The governing equations are the momentum, mass, species, and energy conservation equations. The momentum equation, in dimensionless form, can be written as

\[ d^3F/d\xi^3 + F(d^2F/d\xi^2) = 0, \]

where

\[ dF/d\xi = u/u_*, \]

and

\[ \xi = \left[ \int_0^r \frac{\rho / \rho_\infty}{(2\mu x)/(\rho_\infty u_*)} \right]^{1/2}. \]

The boundary conditions for Eq. (1) are

\[ \frac{dF}{d\xi} = 0 \text{ at } \xi = 0, \]

and

\[ \frac{dF}{d\xi} = 1 \text{ as } \xi \to \infty. \]

The energy/species conservation equation can be written in terms of a coupling function, \( \phi \), as follows (Annamalai and Sibulkin, 1978):

\[ d^2\phi/d\xi^2 + (Sc)F(d\phi/d\xi) = 0. \]

The coupling function in Eq. (6) is defined as

\[ \phi = (\beta - \beta_\infty)/(\beta_\infty - \beta_*), \]

where \( \beta \) is the Schwab–Zeldovich variable which transforms the chemically reacting flow problem into a simple heat and mass transfer problem. For \( Sc = 1.0 \), Eq. (6) becomes

\[ d^2\phi/d\xi^2 + F(d\phi/d\xi) = 0. \]

The boundary conditions are

\[ \phi = 1 \text{ at } \xi = 0, \]

and

\[ \phi = 0 \text{ as } \xi \to \infty. \]

If we let \( F_1 = (1 - dF/d\xi) \), Eqs. (1), (4), and (5) can be rewritten as

\[ d^2F_1/d\xi^2 + F(dF_1/d\xi) = 0, \]

\[ F_1 = 1 \text{ at } \xi = 0, \]

and

\[ F_1 = 0 \text{ as } \xi \to \infty. \]

It is evident that Eqs. (8) and (11), with their corresponding boundary conditions, are mathematically similar. Thus, \( \phi_1 \) and \( F_1 \) are identical functions of \( \xi \).

Since heat has to be transferred to the surface in order to vaporize the fuel, the mass transfer rate is related to the heat transfer rate as

\[ \dot{m}^* = \left[ -\rho DB(d\phi/d\xi) \right]/\left[ (2\mu x)/(\rho_\infty u_*) \right]^{1/2}. \]

The details of the derivation of Eq. (14) are given in Annamalai and Sibulkin (1978). In terms of the Reynolds number, Eq. (14) can be rewritten as

\[ \left[ (\dot{m}^* x)/(\rho DR_e)^{1/2} \right] = -\left\{ B(d\phi/d\xi) \right\}/\sqrt{2}, \]

where \( B \) is the transfer number defined in Annamalai and Sibulkin (1978).

The mass transfer rate, \( \dot{m}^* \), is proportional to the normal velocity at the fuel surface, which is related to \( B \) and \( F \). Thus, Eq. (15) can be rewritten as

\[ (d\phi/d\xi) B = F \text{ at } \xi = 0. \]
2.3 Solution

In the present analytical study, it is desired to solve Eq. (8) subject to the boundary conditions given in Eqs. (9), (10), and (16), using the non-integral method. The resulting derivative \( (d\phi/d\xi) \) at \( \xi = 0 \) is then substituted into Eq. (15) to calculate the burning rate at the fuel surface.

Firstly, the similarity coordinate \( \xi \) is normalized as follows:

\[
\eta = \xi/c,
\]

where the parameter, \( c \), which is to be determined, is chosen so that polynomial profiles may be used directly in Eq. (8) subject to the following boundary conditions at \( \eta = 1 \):

\[
\begin{align*}
f' &= 1 \quad \text{at} \quad \eta = 1, \\
\phi &= 0 \quad \text{at} \quad \eta = 1,
\end{align*}
\]  

(18a)

(18b)

where

\[
f' = (u/u_\infty) = (dF/d\xi).
\]  

(19)

The prime in Eq. (18) and hereon denotes differentiation with respect to \( \eta \).

From Eq. (17), it can be shown that

\[
\frac{d\phi}{d\xi} = \frac{\phi'}{c},
\]

(20)

\[
\frac{d^2\phi}{d\xi^2} = \frac{\phi''}{c^2}.
\]

(21)

Furthermore, integrating Eq. (19) with respect to \( \xi \) and using Eq. (17) result in

\[
F(\xi) = F(0) + c \int_0^\eta f' \, d\eta.
\]

(22)

Substituting Eqs. (20) through (22) into Eq. (8) and replacing \( F(0) \) with \( \phi'(0) \) (from Eq. (16)), one gets

\[
\phi'' + c^2 \left( \phi'(0)/c^2 + \int_0^\eta f' \, d\eta \right) \phi' = 0
\]

(23)

Equation (23) is solved using polynomial profiles for \( \phi \) and \( f' \), which satisfy the corresponding boundary conditions. The objective is to evaluate \( c^2 \) such that the total error in satisfying the differential equation is a minimum. The polynomial profiles for \( \phi \) and \( f' \) are adopted from Annamalai and Sibulkin (1978). They are

\[
\begin{align*}
\phi &= a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3, \\
f' &= c_1 \eta + c_2 \eta^2 + c_3 \eta^3.
\end{align*}
\]

(24)

(25)

and

\[
\begin{align*}
\phi &= a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3, \\
f' &= c_1 \eta + c_2 \eta^2 + c_3 \eta^3,
\end{align*}
\]

where

\[
\begin{align*}
a_0 &= 1, \\
a_1 &= (2/B)(1 - [1 + 3B/2]^{1/2}), \\
a_2 &= -2a_1 - 3, \\
a_3 &= a_1 + 2, \\
c_1 &= -a_1, \\
c_2 &= 2a_1 + 3.
\end{align*}
\]

2.4 Comparison of Results

The present results are now compared with the exact solution which is based on Emmons (1956). By substituting Eq. (27b) into Eq. (29), it can be shown that

\[
\frac{[(m^2)/(\rho D R e_{f,2}^2)]}{B = \pm (a_1) (a/[2(\pm - a_1)(B + 1)])^{1/2},
\]

(29)
By comparing Eq. (31) with the burning rate given in Annamalai and Sibulkin (1978), it is evident that solution A yields an expression for the burning rate which is identical to the integral solution.

Similarly, substituting Eq. (28b) into Eq. (29), an expression for the burning rate for solution B can be obtained. The results from the exact solution and the non-integral solutions A and B are presented in Figure 2. For most of the flammable materials of practical interest, the B number ranges from 0.1 to 7.0 (Annamalai and Sibulkin, 1978). Therefore, the burning rate is presented for the B-numbers between 0.1 and 10.0 only.

Both of the non-integral results compare very well with the exact solution for small values of B, but deviate from the exact solution as B increases. Therefore, the non-integral approximate method predicts the burning rate quite satisfactorily in the case of flow over typical solid fuel surfaces with low values of B (for example, solid fuels and plastics). However, they overestimate the burning rate in the case of flow over liquid fuel surfaces, for which the values of B are generally larger than 3. For large values of B, the non-integral method predicts the burning rate slightly better when the integral of $e$ is set equal to zero (solution A) rather than when the integral of $e^2$ is minimized (solution B).

![FIGURE 2 Variations of burning ratio with transfer number.](image)

where

$$\alpha = (9/70) - (3/140) a_1 - a_1^2/105.$$  \hspace{1cm} (31)

In Eqs. (32) and (33),

$$dF/d\xi = u/u_{ref},$$  \hspace{1cm} (34)

$$\phi = (T - T_w)/(T_e - T_w),$$  \hspace{1cm} (35)

and

$$\xi = (y/x)(Gr_e/4)^{1/4},$$  \hspace{1cm} (36)

where

$$u_{ref} = 2\nu(Gr_e)^{1/2}/x,$$  \hspace{1cm} (37a)

$$Gr_e = g[(T_e - T_w)/T_e]x^2/(\nu^2),$$  \hspace{1cm} (37b)

and $x, y$ are the vertical and horizontal coordinates, respectively.

![FIGURE 3 Schematic of natural convection over vertical plate.](image)
Unlike in the case of forced convection on a flat surface, the derivative $dF/d\xi$ here is defined as $u/u_\infty$. In the present approximate analysis, the reference velocity is dependent on the chosen velocity profile and the conservation equations.

### 3.2 Solution

The non-integral method is now applied to determine the local natural convective heat transfer on the vertical plate. As in the last section, we first normalize the similarity coordinate $\xi$ by letting

$$\eta = (\xi/e) = (y/\eta)(Gr_e/4)^{1/4},$$

so that velocity and temperature profiles which satisfy the appropriate boundary conditions at $\eta = 1$ may be used to solve the momentum and energy equations. We then let

$$f'(\eta) = (dF/d\xi)/d,$$

where the parameter, $d$, takes into account the variation of the reference velocity with the chosen velocity profile. The prime in Eq. (39) again denotes differentiation with respect to $\eta$.

Integrating Eq. (39) with respect to $\xi$ results in

$$F(\xi) = e^{-d}f(\eta).$$

Substituting Eqs. (38), (39), and (40) into Eqs. (32) and (33), and letting

$$e^2d = e_1,$$

and

$$e^2/d = e_2,$$

one gets

$$f'' + 3e_f f' - 2e_1(f')^2 + e_2\phi = 0,$$

$$\phi'' + 3 Pr e_f f\phi' = 0.$$  \hspace{1cm} (44)

The coefficients $e_1$ and $e_2$ in Eqs. (43) and (44) are solved using the non-integral method, for selected polynomial $f'$ and $\phi$ profiles. Once $e_1$ and $e_2$ are obtained, $e$ is evaluated using Eqs. (41) and (42). The local Nusselt number is then determined with the following equation

$$Nu_e = h x/k = [\phi'(0)/e](Gr_e/4)^{1/4},$$

where

$$h = k(\partial T/\partial y)w/(T_\infty - T_\infty).$$

In this study, the local Nusselt number is determined first by using the following velocity and temperature profiles assumed by Eckert and Drake (1959):

$$f' = \eta(1 - \eta)^2,$$

$$\phi = (1 - \eta)^2.$$  \hspace{1cm} (47)

As in the combustion problem, two solutions are obtained by (i) setting the integral of $\epsilon$ between $\eta = 0$ and $\eta = 1$ equal to zero, and by (ii) minimizing the integral of $\epsilon^2$ between the same two limits. The corresponding solutions are

$$Nu_e/Gr_e^{1/4} = 0.508[Pr^2/(0.952 + Pr)]^{1/4},$$

and

$$Nu_e/Gr_e^{1/4} = 0.502[Pr^2/(0.176 + Pr)]^{1/4}.$$  \hspace{1cm} (49)

As in the combustion problem, Eq. (49) is identical to the integral solution (Eckert and Drake, 1959). The procedure for determining the local Nusselt number is repeated using the assumed velocity profile given in Eq. (47), but replacing the second order temperature profile with a third-order polynomial profile as follows:

$$\phi = 1 - 3\eta/2 + \eta^2/2.$$  \hspace{1cm} (51)

The temperature profile given in Eq. (51) satisfies the energy conservation Eq. (44) identically at $\eta = 0$. The solutions corresponding to setting the integral of $\epsilon$ equal to zero and minimizing the integral of $\epsilon^2$ are

$$Nu_e/Gr_e^{1/4} = 0.436[Pr^2/(0.625 + Pr)]^{1/4},$$

and

$$Nu_e/Gr_e^{1/4} = 0.419[Pr^2/(0.141 + Pr)]^{1/4}.$$  \hspace{1cm} (52)

It can be shown that Eq. (52) is identical to the integral solution using the $f'$ and $\phi$ profiles given in Eqs. (47) and (51).

### 3.3 Comparison of Results

Equations (49), (50), (52), and (53) all have the same functional form and give $Nu_e/Gr_e^{1/4}$ as functions of $Pr$. The various solutions are presented in Figure 4.
along with the numerical solution of Eqs. (32) and (33) subject to the appropriate boundary conditions (Kays and Crawford, 1980).

From Figure 4, it is apparent that Eq. (49) correlates best with the numerical result over the range of Prandtl numbers plotted, with a maximum error of about 10 percent at $Pr = 0.01$. Equation (50) is more accurate than Eq. (49) at high Prandtl numbers. As the Prandtl number approaches infinity, Eq. (50) becomes

$$\left(\frac{Nu_s}{Gr_s^{1/4}}\right) = 0.502 Pr^{1/4},$$

which is in excellent agreement with the following exact solution given in Kays and Crawford (1980) as $Pr \to \infty$:

$$\left(\frac{Nu_s}{Gr_s^{1/4}}\right) = 0.503 Pr^{1/4}. \tag{55}$$

However, the curve based on Eq. (50) deviates from that based on the numerical solution at low Prandtl numbers. Equation (52) underpredicts the local heat transfer over the entire range $0.01 \leq Pr \leq 1,000$. Finally, the curve based on Eq. (53) is too high at low Prandtl numbers and too low at high Prandtl numbers.

4. CLOSURE

The application of the non-integral method to solve for the burning rate of a fuel surface subject to forced convection conditions, and the natural convective heat transfer from a vertical isothermal surface is demonstrated. In each case, two solutions are obtained by substituting assumed polynomial profiles for the dependent variables into the appropriate governing differential equations. In the first solution, the total error in satisfying the governing equations is set equal to zero, whereas, in the second solution, the total of the square of the errors is minimized. Both non-integral solutions compare favorably with the numerical solutions, and are found to be sensitive to the selection of the assumed profiles.

It has been shown that the first of the two non-integral solutions yields results which are identical to the corresponding integral solutions in both cases studied. The identity of the first non-integral solution and the corresponding integral solution has been further verified by the authors, who applied the non-integral method (by setting the total error equal to zero) to the burning of a vertical carbon surface subject to natural convection (whose integral solution has been given in Annamalai et al. 1986b) and the various problems examined previously in Annamalai et al. (1986a). Compared to the integral method, the main advantages of the present non-integral method are: (a) once the assumed profiles are substituted into the governing differential equations, the resulting equations are algebraic equations, and (b) the integration of algebraic equations can be performed readily with available computer software.

**NOMENCLATURE**

$a_0$, $a_1$, $a_2$, $a_3$ coefficients in the polynomial for $\phi$, Eq. (24)

$B$ transfer number (Annamalai and Sibulkin, 1978)

**APPLICATIONS OF NON-INTEGRAL METHOD**

dimensionless parameter, Eq. (17)

coefficients in the polynomial for $f'$, Eq. (25)

diffusion coefficient

dimensionless parameter, Eq. (39)

dimensionless velocity, Eqs. (2) and (34)

dimensionless parameter, Eq. (38)

dimensional parameters, Eqs. (41) and (42)

$(1 - dF/d\xi)$

dimensionless velocity function of $\eta$

Grashof number, Eq. (37b)

gravitational acceleration

convective heat transfer coefficient

thermal conductivity

mass burning rate per unit area

local Nusselt number

$Pr$

Prandtl number

$Re_s$

Reynolds number based on $x$

$Sc$

Schmidt number

$T$

temperature

$u$

velocity along $x$

$x$

coordinate along the main flow

$y$

coordinate normal to the main flow

$\beta$

Schwab–Zeldovich variable

$\delta$

boundary layer thickness

$\varepsilon$

error

$\eta$

similarity parameter used in polynomial profiles, $y/\delta$

$\mu$

dynamic viscosity

$\nu$

kinematic viscosity

conventional similarity variable, Eqs. (3) and (36)

$\rho$

density

$\phi$

normalized coupling function for the combustion problem, and

dimensionless temperature for the free convection problem, Eqs. (7) and (35)

**Subscripts**

$ref$

reference

$w$

wall

$\infty$

free stream
REFERENCES


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